ON THE REALIZATION OF SYMMETRIES IN QUANTUM MECHANICS

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Abstract. The aim of this paper is to give a simple, geometric proof of Wigner's theorem on the realization of symmetries in quantum mechanics that clarifies its relation to projective geometry. Although several proofs exist already, it seems that the relevance of Wigner's theorem is not fully appreciated in general. It is Wigner's theorem which allows the use of linear realizations of symmetries and therefore guarantees that, in the end, quantum theory stays a linear theory. In the present paper, we take a strictly geometrical point of view in order to prove this theorem. It becomes apparent that Wigner's theorem is nothing else but a corollary of the fundamental theorem of projective geometry. In this sense, the proof presented here is simple, transparent and therefore accessible even to elementary treatments in quantum mechanics.

1. Introduction

There is no doubt that symmetries play a very important role in physics. For any given problem the existence of symmetries is decisive and their realization is crucial when it comes to the solution of the problem. How symmetries are realized depends of course on the theory under consideration and more precisely on the corresponding structure of its space of states. It is well known that in quantum mechanics symmetries are realized by special linear or anti-linear operators. We expect this since the Hilbert space is linear. In this consideration, however, we overlook the fact that, because of the probability interpretation of the wave function, a state is not a single vector but a ray in Hilbert space, i.e. a one-dimensional subspace. Consequently, the space of quantum mechanical states is not a linear but a projective space. Since projective spaces are not linear (and hence the space of states is not linear) the linear realization of symmetries cannot be taken for granted. On the contrary we may doubt if such a linear realization is possible at all. It is due to the work of E. Wigner and his almost forgotten theorem on the realization of symmetries, a theorem disregarded even by many otherwise excellent and generally accepted books and lectures on quantum mechanics, that we may nevertheless always use linear or anti-linear realizations in quantum mechanics. It states that

> *Any symmetry transformation on the set of pure states of a quantum mechanical system is represented up to a scalar factor by either a unitary or an anti-unitary transformation on the corresponding Hilbert space.*

By symmetry transformation we mean here a bijective map, a ray transformation which preserves transition probabilities.

We intend to discuss this so often ignored theorem and to give an elementary and simple proof, which relies on its geometrical background. This background of course is dominated by projective geometry. Despite its non linear nature it is projective geometry that gives a deeper justification for the usage of linear or anti-linear realizations in quantum mechanics and consequently in quantum field theory. In the literature one can find a variety of different proofs of Wigner's theorem: [13, 6, 10, 5, 9, 2, 11, 3, 12, 4] and there are some that make a connection to projective geometry:

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[10, 5, 9, 11]. However, almost all of them make no direct use of it, and where they do the key concept is covered by a complex presentation or by a generalizing idea in one or the other direction. The aim of this paper is to give a simple geometric proof of Wigner's original theorem without any generalization whatsoever, and as a consequence to clarify the key role projective geometry has when it comes to the representation of symmetries within quantum physics.

In this paper the proof of Wigner's theorem is a direct consequence of projective geometry: Wigner's theorem is essentially a corollary of the fundamental theorem of projective geometry. In this sense, Wigner's theorem could be treated in an elementary way in any quantum mechanics textbook and subsequently in any lecture.

In the following we will first give a short review of the geometric proof of Wigner's theorem. The proof contains four steps. We first show that a symmetry transformation fulfills the premises of the fundamental theorem of projective geometry: any symmetry transformation is a collineation, i.e. it maps projective lines to projective lines. In the second step we use the fundamental theorem of projective geometry in the form and proof given by E. Artin [1]: any collineation between finite dimensional projective spaces over an arbitrary field $\mathbb{K}$ is a semi-projectivity. In this paper we use only $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, denoting either the field of real or the field of complex numbers. By semi-projectivity we mean a bijective map between projective spaces which is induced by a bijective linear or anti-linear map between the associated vector spaces. Observe that the fundamental theorem of projective geometry refers to (finite-dimensional) $\mathbb{K}$-vector spaces with no further structure and not to Hilbert spaces so that we cannot talk about unitarity here. In the third step we take a Hilbert space instead of a vector space and we consider the corresponding formulation as established in Artin's proof. In this way we can show that the linear operator is a unitary and the anti-linear operator is an anti-unitary map. In the last step the extension to infinite dimensions is performed. One can imagine that the difficulty is to prove the existence of a ray consistent semi-linear transformation, which is done in Artin's proof in the second step. What the other steps are concerned with is quite easy to prove as soon as you have realized the existence of the corresponding statements.

The remainder of this paper is organized as follows: in the subsequent section (2) we discuss the basic concepts of projective geometry and introduce its fundamental theorem. We refer very shortly to the first proof given by F. Klein [8], who also used concepts found by A. F. Möbius, but for our treatment we rely mainly on the work of E. Artin. We summarize briefly the steps of his proof since we find it useful for the understanding of the topic. We will argue how this is related to Wigner's theorem in section 3. In section 4 we give our proof of Wigner's theorem, which is, within the framework of projective geometry, simple and elementary. The key points are summarized and a short conclusion is given in the last section (5).

2. On the fundamental theorem of projective geometry

Before we start with the discussion of Wigner's theorem and its geometric proof, we use this preliminary section to clarify the notation used throughout the paper.

A *ray* A in projective geometry is an orbit $[a]$ of the group of units $\mathbb{K}^\times$ on a vector space V over the field $\mathbb{K}$, i.e. A is a one-dimensional subspace of V (with the zero element removed). We write $[a]$ for the ray with representative $a \in V$. The set of all such rays in V is called *projective space* $\mathcal{P}V$. Its dimension as a manifold is

$$\dim \mathcal{P}V = \dim V - 1\,. \tag{1}$$

Some of the properties of vectors in V are preserved as we go to the projection $\mathcal{P}V$. We call a set $\{\mathsf{A}_1, \ldots, \mathsf{A}_n\}$ of two or more rays *projectively independent*, if and only if there is a linearly independent set of vectors $\{a_1, \ldots, a_n\}$ such that $\forall k \in \{1, \ldots, n\} : a_k \in \mathsf{A}_k$. Observe that two rays are either projectively independent or equal. In a projective space $\mathcal{P}V$ of dimension n a set $\mathcal{B} \equiv \{\mathsf{B}_1, \ldots, \mathsf{B}_{n+2}\} \subset \mathcal{P}V$ of $n+2$ rays is called a projective base of $\mathcal{P}V$, if and only if any subset of $\mathcal{B}$ containing $n+1$ rays is projectively independent.

Regarding two different rays $\mathsf{A}, \mathsf{B} \in \mathcal{P}V$, there is a natural operation, the *unification*, defined by:

$$\mathsf{A} \vee \mathsf{B} := \{[a+b] : \, a \in \mathsf{A}, \, b \in \mathsf{B}\}\ . \tag{2}$$

$\mathsf{A} \vee \mathsf{B}$ is the plane spanned by the projectively independent rays A and B. Since $\dim(\mathsf{A} \vee \mathsf{B}) = 1$ it is natural to refer to this unification as the *projective line* uniquely determined by the projective

points A and B. This gives rise to another notion used in projective geometry: three or more distinct points $\mathsf{A}_1, \ldots, \mathsf{A}_n$ are called *collinear* if and only if they are on the same projective line, i.e. $\forall k \in \{1, \ldots, n\} : \mathsf{A}_k \in \mathsf{A}_1 \vee \mathsf{A}_2$.

With the notions we introduced above, we are now able to define the basic maps between projective spaces which are the collineation and the semi-projectivity. A *collineation* $\mathbf{f} : \mathcal{P}V \to \mathcal{P}W$ is a bijective map that preserves collinearity, i.e. $\mathbf{f}$ maps any projective line to a projective line:

$$\mathbf{f}\left(\mathsf{A} \vee \mathsf{B}\right) = \mathbf{f}\left(\mathsf{A}\right) \vee \mathbf{f}\left(\mathsf{B}\right) . \tag{3}$$

A *semi-projectivity* $\mathbf{g} : \mathcal{P}V \to \mathcal{P}W$ on the other hand is a bijective map that is induced by a semi-linear map $G : V \to W$, i.e.

$$[Ga] = \mathbf{g}\left([a]\right) . \tag{4}$$

Any map G that fulfills equation (4) is called *compatible* with $\mathbf{g}$. The prefix "semi" stands for "up to a field automorphism". Hence the semi-linear map G is linear up to a field automorphism $\sigma : \mathbb{K} \to \mathbb{K}$, i.e. $\forall \alpha, \beta \in \mathbb{K}$, $\forall a, b \in V$:

$$G\left(\alpha a + \beta b\right) = \sigma\left(\alpha\right)\, Ga + \sigma\left(\beta\right)\, Gb\,. \tag{5}$$

Later it will become apparent that the only field automorphisms occuring in the context of Wigner's theorem are either the identity or complex conjugation. Hence, the map G is either linear or anti-linear. Obviously, any semi-projectivity preserves collinearity, i.e. is a collineation. The reverse statement is also true, but not trivial at all, it is the *fundamental theorem of projective geometry*:

Let $\mathcal{P}V$ and $\mathcal{P}W$ be projective spaces of same dimension $n \geq 2$. Then any collineation $\mathbf{f} : \mathcal{P}V \to \mathcal{P}W$ is a semi-projectivity, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, dashed, "F"] \arrow[d] & W \arrow[d] \\ \mathcal{P}V \arrow[r, "\mathbf{f}"] & \mathcal{P}W \end{tikzcd}$$

where $\mathbf{f}$ is a collineation and F is semi-linear.

There are essentially two ways to prove this theorem, the first of which was carried out by F. Klein in 1925 [8]. Klein explicitly shows the assertion for the real projective plane but his construction can be generalized to real projective spaces of arbitrary (finite) dimension (for a more detailed discussion consult for example the corresponding section (I.2) of reference [7]). Out of a projective base and by using only the properties of the collineation he constructs a dense set of intersection points of projective lines. Hence there is only one continuous collineation that maps the projective base to the corresponding image base. But the base and its image also determine a projectivity, and Klein concludes that the regarded collineation is a projectivity by the fact that any projectivity also is a collineation.

The second proof was introduced by E. Artin in 1957 [1] and is valid for projective spaces of (finite) dimension greater or equal two over an arbitrary field. The intention is to show that for every collineation $\mathbf{f} : \mathcal{P}V \to \mathcal{P}W$ there exists a semi-linear transformation $F : V \to W$ which is compatible with $\mathbf{f}$. This means that for a given basis $\{v_k\}_{k\in\{0,\ldots,n\}}$ of V there exists a basis $\{w_k\}_{k\in\{0,\ldots,n\}}$ of W and a semi-linear transformation F such that $\forall k \in \{0, \ldots, n\}$: $w_k = F\left(v_k\right)$. Due to semi-linearity, a given field automorphism $\sigma : \mathbb{K} \to \mathbb{K}$, F then obviously fulfills $\forall \lambda_k \in \mathbb{K}$: $F\left(\lambda_0 v_0 + \cdots + \lambda_n v_n\right) = \sigma\left(\lambda_0\right)\, w_0 + \cdots + \sigma\left(\lambda_n\right)\, w_n$. Furthermore compatibility requires:

$$\mathbf{f}\left[\lambda_0 v_0 + \cdots + \lambda_n v_n\right] = \left[\sigma\left(\lambda_0\right)\, w_0 + \cdots + \sigma\left(\lambda_n\right)\, w_n\right] . \tag{6}$$

In other words, the image of a representative of a ray in V given by a linear combination of the basis vectors and expressed with the "same" (up to a field automorphism) linear combination of the corresponding basis vectors in W, is a representative of the image ray, given above. Artin's proof of this assertion is divided into several steps. As one expects, collinearity is used in almost all of them. Additionally, he uses induction with respect to the dimensional parameter n in order to generalize his construction to spaces of arbitrary (finite) dimension. In what follows we review shortly some of the steps since we believe that this contributes essentially to the understanding of the topic.

(1) Using induction with respect to the dimension allows to show that $\mathbf{f}$ conserves not only projective lines, but also projective subspaces of arbitrary, finite dimension, $\forall n \in \mathbb{N}$:

$$\mathbf{f}\left(\mathsf{A}_0 \vee \mathsf{A}_1 \vee \cdots \vee \mathsf{A}_n\right) = \mathbf{f}\left(\mathsf{A}_0\right) \vee \mathbf{f}\left(\mathsf{A}_1\right) \vee \cdots \vee \mathbf{f}\left(\mathsf{A}_n\right) . \tag{7}$$

(2) For every basis $\{v_k\}_{k\in\{0,\ldots,n\}}$ in V there exists a basis $\{w_k\}_{k\in\{0,\ldots,n\}}$ in W such that $\forall k \in \{0, \ldots, n\}$:

$$\mathbf{f}\left[v_k\right] = \left[w_k\right] \qquad \text{and} \qquad \mathbf{f}\left[v_0 + v_k\right] = \left[w_0 + w_k\right] . \tag{8}$$

(3) There exists a $\mathbb{K}$-automorphism $\sigma : \mathbb{K} \to \mathbb{K}$, such that $\forall k \in \{1, \ldots, n\}$, $\lambda \in \mathbb{K}$:

$$\mathbf{f}\left[v_0 + \lambda v_k\right] = \left[w_0 + \sigma\left(\lambda\right) w_k\right] . \tag{9}$$

(4) Using again induction, one can show that:

$$\mathbf{f}\left[v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n\right] = \left[w_0 + \sigma\left(\lambda_1\right)\, w_1 + \cdots + \sigma\left(\lambda_n\right)\, w_n\right] . \tag{10}$$

(5) This statement, by using the fact that σ is a field automorphism, finally leads to

$$\mathbf{f}\left[\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n\right] = \left[\sigma\left(\lambda_0\right)\, w_0 + \sigma\left(\lambda_1\right)\, w_1 + \cdots + \sigma\left(\lambda_n\right)\, w_n\right] , \tag{11}$$

which completes Artin's proof of the fundamental theorem of projective geometry.

Let us just make one small remark on the way the field automorphism $\sigma : \mathbb{K} \to \mathbb{K}$ that we have not found in the proof by F. Klein enters in the third step of Artin's proof. To be a bit more precise, Artin introduces maps

$$\begin{array}{ccc} \lambda & \mapsto & [v_0 + \lambda v_k] \\ \mathbb{K} & \to & [v_0] \vee [v_k] \end{array}$$

which are just one way to write the homeomorphisms between the projective lines $[v_0]\vee[v_k]$ and the one-point compactification $\mathbb{K} \cup \{\infty\}$ of the field $\mathbb{K}$. He gets seemingly different maps $\sigma_k : \mathbb{K} \to \mathbb{K}$ by applying the following diagram

$$\begin{tikzcd} \mathbb{K} \cup \{\infty\} \arrow[r, dashed, "\sigma_i"] \arrow[d, hook] & \mathbb{K} \cup \{\infty\} \\ {[v_0] \vee [v_i]} \arrow[r, hook, two heads, "\mathbf{f}"] & {[w_0] \vee [w_i]} \arrow[u, hook] \end{tikzcd}$$

But the maps σ_k can then be shown to be all the same, unique field automorphism $\sigma : \mathbb{K} \to \mathbb{K}$.

3. Symmetries and Wigner's theorem

It is well known that the set of pure states of a quantum mechanical system may be described by the set of one-dimensional subspaces of the corresponding Hilbert space, i.e. the projective Hilbert space.

If we define the projective space for some Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$, we find an additional structure on $\mathcal{PH}$, stemming from the scalar product $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ on $\mathcal{H}$. A natural definition for this structure is:

$$\mathsf{A} \odot \mathsf{B} := \frac{|\langle a|b\rangle|^2}{\|a\|^2\, \|b\|^2} \quad \forall a \in \mathsf{A},\, b \in \mathsf{B} . \tag{12}$$

This is the transition probability between states A and B. Observe that if $\mathsf{A} \odot \mathsf{B} = 0$ any representative of A is orthogonal to any representative of B. On the other hand, if $\mathsf{A} \odot \mathsf{B} = 1$ then, by Schwarz's inequality $\mathsf{A} = \mathsf{B}$.

We define the *projective Hilbert space* to be the pair $(\mathcal{PH}, \odot)$, where $\mathcal{PH}$ is the ordinary projective space corresponding to the vector space structure and $\odot : \mathcal{PH} \times \mathcal{PH} \to [0, 1]$ is the function induced by the scalar product, defined as above.

A symmetry transformation is a map on the space of states that preserves transition probabilities. Hence, within our notation a symmetry transformation $\mathbf{T}$ is a bijective map $\mathbf{T} : \mathcal{PH} \to \mathcal{PH}'$ such that $\forall \mathsf{A}, \mathsf{B} \in \mathcal{PH}$:

$$\mathsf{A} \odot \mathsf{B} = \mathbf{T}\mathsf{A} \odot \mathbf{T}\mathsf{B} . \tag{13}$$

Obviously, by this equation $\mathbf{T}$ preserves orthogonality, and this is why we may also call this property of the symmetry transformation $\mathbf{T}$ *quasi-unitarity*.

With the help of these refined notions we can restate Wigner's theorem in a more mathematical way.

Let $\mathcal{P}V$ and $\mathcal{P}W$ be projective spaces of same dimension $n \geq 2$. Then any collineation $\mathbf{f} : \mathcal{P}V \to \mathcal{P}W$ is a semi-projectivity, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, dashed, "F"] \arrow[d, two heads] & W \arrow[d, two heads] \\ \mathcal{P}V \arrow[r, "\mathbf{f}"] & \mathcal{P}W \end{tikzcd}$$

where $\mathbf{f}$ is a collineation and F is semi-linear.

Let $\mathbf{T} : \mathcal{PH} \to \mathcal{PH}'$ be a symmetry (i.e. quasi-unitary) transformation, then there exists a compatible semi-unitary transformation $U : \mathcal{H} \to \mathcal{H}'$, i.e. $\forall a \in \mathcal{H}$:

$$\mathbf{T}\,[a] = [Ua]$$

and hence the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[r, dashed, "U"] \arrow[d, two heads] & \mathcal{H}' \arrow[d, two heads] \\ \mathcal{PH} \arrow[r, "\mathbf{T}"] & \mathcal{PH}' \end{tikzcd}$$

What we encounter here is obviously a lifting problem. Observe the similarity to the fundamental theorem of projective geometry. It is exactly this similarity that we will use in order to formulate a geometric proof of Wigner's theorem. Despite this observation, it is not a priori clear that this apparent similarity indeed leads to a geometric proof of Wigner's theorem, which, as we will see in the following section, it does.

We know that the set of pure states is divided into disjoint super selection sectors. Due to the super selection rules states of different super selection sectors cannot be superposed. Nevertheless a symmetry may be a map from one of these sectors to another. In order to be able to formulate and prove the theorem without further difficulties, the regarded projective Hilbert spaces should be two (or even the same) super selection sectors, in which the superposition principle holds without limitations.

4. Geometric proof of Wigner's theorem

The proof we present here consists of four steps. In the first step we will show that any quantum symmetry transformation is also a projective collineation. This is the key that makes it possible to apply the fundamental theorem of projective geometry. In the second step, using the fundamental theorem of projective geometry, we show that any symmetry transformation (which is, as shown in step 1 below, a collineation) is induced by a semi-linear transformation. In step 3 we show that this semi-linear transformation is actually semi-unitary. This completes our proof of Wigner's theorem for finite dimensional Hilbert spaces. In the last step we extend this result to Hilbert spaces of infinite dimension.

Step 1.

Any symmetry (quasi-unitary) transformation is a collineation.

We have to show that

$$\mathbf{T}\,(\mathsf{A} \vee \mathsf{B}) = \mathbf{T}\mathsf{A} \vee \mathbf{T}\mathsf{B} \tag{14}$$

for any quasi-unitary transformation $\mathbf{T}$. To prove this, we take a projective line $\mathsf{A} \vee \mathsf{B}$ and choose the rays A and B to be orthogonal. Then there is an orthogonal base (OGB) $\{b_k\}_{k\in I}$ of $\mathcal{H}$, such that

$$[b_1] = \mathsf{A} \quad \text{and} \quad [b_2] = \mathsf{B}\,. \tag{15}$$

In this base we can write any representative c of any ray $\mathsf{C} \in \mathsf{A} \vee \mathsf{B}$ as

$$c = \gamma_1 b_1 + \gamma_2 b_2\,. \tag{16}$$

Obviously $\mathsf{A} \odot \mathsf{B} = 0$, and because $\mathbf{T}$ is quasi-unitary we also have

$$\mathbf{T}\mathsf{A} \odot \mathbf{T}\mathsf{B} = 0\,. \tag{17}$$

Furthermore $\mathbf{T}$ maps the orthogonal rays $\{[b_k]\}_{k\in I}$ onto orthogonal rays $\{[b'_k]\}_{k\in I}$. This is elementary since for $\mathsf{B}_k = [b_k]$ and $\mathsf{B}'_k = [b'_k]$ we have

$$\mathsf{B}'_k \odot \mathsf{B}'_l = \mathbf{T}\mathsf{B}_k \odot \mathbf{T}\mathsf{B}_l = \mathsf{B}_k \odot \mathsf{B}_l = \delta_{kl}\,. \tag{18}$$

Since $\mathbf{T}$ is bijective, any set of representatives $\{b'_k\}_{k\in I}$ of these orthogonal rays $\{\mathsf{B}'_k\}_{k\in I}$ is an orthogonal base of $\mathcal{H}'$. Hence we may write any representative c' of $\mathbf{T}\mathsf{C} \in \mathbf{T}\,(\mathsf{A} \vee \mathsf{B})$ as

$$c' = \sum_{k\in I} \gamma'_k\, b'_k\,. \tag{19}$$

But for the coefficients γ'_k we have

$$\frac{\|b'_k\|^2}{\|c'\|^2}\,|\gamma'_k|^2 = \frac{|\langle b'_k|c'\rangle|^2}{\|b'_k\|^2\,\|c'\|^2} = \mathbf{T}\,[b_k] \odot \mathbf{T}\,[c] = [b_k] \odot [c] = \frac{|\langle b_k|c\rangle|^2}{\|b_k\|^2\,\|c\|^2} = \frac{\|b_k\|^2}{\|c\|^2}\,|\gamma_k|^2\,, \tag{20}$$

and hence $\gamma'_k = 0\ \forall k \geq 3$. Equation (19) then reduces to

$$c' = \gamma'_1\, b'_1 + \gamma'_2\, b'_2\,. \tag{21}$$

This means that any representative of $\mathbf{T}\mathsf{C} \in \mathbf{T}\,(\mathsf{A} \vee \mathsf{B})$ is an element of the plane spanned by the rays $[b'_1] = \mathbf{T}\mathsf{A}$ and $[b'_2] = \mathbf{T}\mathsf{B}$, i.e. the projective line $\mathbf{T}\mathsf{A} \vee \mathbf{T}\mathsf{B}$.

Hence it follows that

$$\mathbf{T}\,(\mathsf{A} \vee \mathsf{B}) \subset \mathbf{T}\mathsf{A} \vee \mathbf{T}\mathsf{B}\,. \tag{22}$$

And again since $\mathbf{T}$ is bijective one easily verifies that:

$$\mathbf{T}\,(\mathsf{A} \vee \mathsf{B}) = \mathbf{T}\mathsf{A} \vee \mathbf{T}\mathsf{B}\,. \tag{23}$$

Thus any quasi-unitary transformation $\mathbf{T}$ is a collineation.

Step 2. Since we know that $\mathbf{T}$ is a collineation, we can apply the fundamental theorem of projective geometry (p. 5). It then follows that $\mathbf{T}$ is a semi-projectivity. This means that $\mathbf{T}$ is induced by either a linear or an anti-linear transformation U between the $\mathbb{C}$-vector spaces $\mathcal{H}$ and $\mathcal{H}'$, which conversely is compatible with the ray transformation $\mathbf{T}$:

Any symmetry (quasi-unitary) transformation $\mathbf{T}$
is induced by a semi-linear transformation U*:*

$$\begin{tikzcd} \mathcal{H} \arrow[r, dashed, "U", "\text{semi-linear}"'] \arrow[d] & \mathcal{H}' \arrow[d] \\ \mathcal{PH} \arrow[r, "\mathbf{T}"] & \mathcal{PH}' \end{tikzcd}$$

Observe that in $\mathbb{C}$ there are only two field automorphisms mapping the unit element to itself: the identity and the complex conjugation. Apparently, regarding equations (8) and (9), the field automorphism σ fulfills $\sigma(1) = 1$. Hence U is either linear or anti-linear.

Step 3. Now, as implicitly in the first step, too, we consider $\mathcal{H}$ as a Hilbert space. So in addition we dispose of the standard Hermitian scalar product. The symmetry transformation $\mathbf{T}$ conserves probabilities by assumption:

$$\mathbf{T}\mathsf{A} \odot \mathbf{T}\mathsf{B} = \mathsf{A} \odot \mathsf{B}\,.$$

Then, by compatibility, the semi-linear transformation U also respects transition probabilities. Additionally it maps orthogonal vectors onto orthogonal vectors. This is enough to demand for U mapping one ONB to another ONB. More explicitly, we have:

$$[b'_1 + b'_k] \odot [b'_1] = [b_1 + b_k] \odot [b_1] \tag{24}$$

From which by $\frac{\|b_1+b_k\|^2\|b_1\|^2}{|\langle b_1+b_k|b_1\rangle|^2} = \frac{\left(\|b_1\|^2+\|b_k\|^2\right)\|b_1\|^2}{\|b_1\|^4} = 1 + \frac{\|b_k\|^2}{\|b_1\|^2}$ it follows that

$$\frac{\|b_k'\|^2}{\|b_1'\|^2} = \frac{\|b_k\|^2}{\|b_1\|^2}\,. \tag{25}$$

Hence if we choose $\{b_k\}_{k\in I}$ to be an ONB of $\mathcal{H}$, we can choose $\|b_1'\| = 1$ and obtain another ONB $\{b_k'\}_{k\in I}$ of $\mathcal{H}'$. Then, since any semi-linear transformation is semi-unitary, if it maps one ONB to another, we conclude that the map U, defined by

$$Ub_k := b_k' \quad \forall k \in I \tag{26}$$

is semi-unitary. This completes our proof of Wigner's theorem for a finite dimensional Hilbert space.

We already know from Artin's proof that the image base $\{b_k'\}_{k\in I}$ is determined up to one single overall factor, so, since we already asked for $\|b_1'\|$ to be one, there's one last degree of freedom left in the choice of the semi-unitary map U: the semi-unitary map U is determined up to a phase factor.

We have shown Wigner's theorem in a geometric way that emphasizes and clarifies the connection to the fundamental theorem of projective geometry at least for finite index sets I. Observe that the fundamental theorem of projective geometry is a theorem that deals with (finite dimensional) $\mathbb{K}$-vector spaces. In order to obtain Wigner's theorem for an infinite dimensional Hilbert space, questions of convergence have to be examined. Notice however that steps 1 and 3 are valid also for infinite dimensions.

Step 4. Now $\mathcal{H}$ and $\mathcal{H}'$ are infinite dimensional spaces. In this case we assume of course that $\mathbf{T}$ is continuous. Taking the bases $\{b_k\}_{k\in I}$ and $\{b_k'\}_{k\in I}$ of step 3 as countable bases for some infinite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively, we can define a semi-unitary operator $U : \mathcal{H} \to \mathcal{H}'$ by:

$$Ub_k := b_k' \quad \forall k \in I\,. \tag{27}$$

Regardless of the fact that we already used the letter U to denote the semi-unitary, compatible map in the case of finite dimensional Hilbert spaces, we use the same letter U to introduce a semi-unitary map between Hilbert spaces of infinite dimension. The goal of this last step is to show that the operator U, defined by (27) is, even in the case of infinite dimensions, compatible with the quantum symmetry transformation $\mathbf{T} : \mathcal{PH} \to \mathcal{PH}'$, as the commutative diagram indicates:

$$\begin{tikzcd} \mathcal{H} \arrow[r, dashed, "U"] \arrow[d] & \mathcal{H}' \arrow[d] \\ \mathcal{PH} \arrow[r, "\mathbf{T}"] & \mathcal{PH}' \end{tikzcd}$$

This means, we have to show that $\forall \alpha_k \in \mathbb{C}$:

$$\mathbf{T}\left[\sum_{k=1}^{\infty} \alpha_k\, b_k\right] = \left[\sum_{k=1}^{\infty} \sigma\left(\alpha_k\right)\, Ub_k\right]\,. \tag{28}$$

In steps 1-3 we already have proven Wigner's theorem for finite dimensions, i.e. $\forall n \in \mathbb{N}$:

$$\mathbf{T}\left[\sum_{k=1}^{n} \alpha_k\, b_k\right] = \left[\sum_{k=1}^{n} \sigma\left(\alpha_k\right)\, Ub_k\right]\,. \tag{29}$$

Since the projection $\pi : \mathcal{H} \ni x \mapsto [x] \in \mathcal{PH}$ is by definition continuous (with respect to the quotient topology on $\mathcal{PH}$) we have for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$:

$$\left[\lim_{n\to\infty} x_n\right] = \lim_{n\to\infty} \left[x_n\right]\,, \tag{30}$$

Hence, since the quantum symmetry $\mathbf{T}$ is also a continuous transformation, we obtain directly:

$$\begin{aligned} \lim_{n\to\infty} \mathbf{T}\left[\sum_{k=1}^{n} \alpha_k\, b_k\right] &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \sigma\left(\alpha_k\right)\, Ub_k\right] \\ \mathbf{T}\left[\lim_{n\to\infty}\sum_{k=1}^{n} \alpha_k\, b_k\right] &= \left[\lim_{n\to\infty}\sum_{k=1}^{n} \sigma\left(\alpha_k\right)\, Ub_k\right]\,, \end{aligned} \tag{31}$$

which is the assertion (28), now proven to be correct. This completes the proof of Wigner's theorem also for infinite dimensions. We see that our derivation is geometric, since it results with the help of some basic considerations directly from the fundamental theorem of projective geometry.

5. Conclusions

Wigner's theorem on the realization of symmetries in quantum mechanics is a result of fundamental importance in Physics. Although different proofs of this theorem are available in the literature, to the best of our knowledge non of them emphasizes the close relation that exists between Wigner's theorem and the fundamental theorem of projective geometry. In this paper, we have adopted a strictly geometric point of view in order to prove Wigner's theorem. Using the tools of projective geometry, suitably adapted to the case of infinite dimensions, Wigner's theorem becomes a corollary of the fundamental theorem of projective geometry. In particular, we have proved that any symmetry transformation is a projective collineation and hence (by the fundamental theorem of projective geometry) must also be a semi-projectivity, i.e. it is induced by a semi-linear (linear or anti-linear) map on the Hilbert space. Using the fact that symmetry transformations preserve, by definition, transition probabilities, we have also proved that this semi-linear map is in fact either a unitary or an anti-unitary operator. As stated in the introduction, there are some proofs that mention the relation to projective geometry. But the close connection to the fundamental theorem of projective geometry is obscured by the parallel use of other concepts. In reference [11], projective geometry appears as a structure naturally contained in a system of propositions describing a quantum system. In that context, symmetries are interpreted as automorphisms of the proposition structure, and a link to Wigner's theorem can be obtained, though not a direct one, in the sense that many other, rather difficult, concepts are involved. A similar remark applies to [5]. In [9], the similarity between Wigner's theorem and Artin's proof [1] of the fundamental theorem of projective geometry is mentioned, but not fully exploited. In contrast, in our analysis we use the tools of projective geometry throughout. This has enabled us to obtain Wigner's theorem in a clear and concise way as a consequence of the fundamental theorem of projective geometry.

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